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Computation of Eigenvalues for Second-Order Differential Equations Using Imbedding Techniques[1]

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ABSTRACT

The invariant imbedding method is used to provide algorithms for numerically computing eigenvalues for the Schrödinger equation and other eigenvalue problems. For the sake of illustration, the Hermite and Bessel equations are studied regarding their eigenvalues and, in the Hermite case, the asymptotic behavior of the eigenfunctions. The most salient feature is that the algorithms provided have good computational behavior in dealing with notoriously unstable problems.

I. INTRODUCTION

In recent papers Wing [1], Beyer [2], and Hagin [3] have applied the method of invariant imbedding to the study of the asymptotic behavior of solutions to certain initial value problems. Specifically, the behavior for large t of solutions to the following type of problems were studied:

$$u''(t) + (c + g(t))\,u(t) = 0, \qquad u(x) = \cos\theta, \qquad u'(x) = \sin\theta \tag{1}$$

where g is "small" in some sense as $t \to \infty$.

It was pointed out in [3] that the techniques used apparently offered a workable algorithm for making parameter studies (c the parameter). For example, certain eigenvalue problems associated with (1) were suggested.

Here we primarily wish to accomplish two things. First, we show that the imbedding technique is applicable to a much larger class of problems than those studied before (which were, roughly speaking, perturbations of constant coefficient equations for large t). Secondly, we emphasize the computational features of the algorithms provided by the imbedding process (the main feature can be sum-

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marized as "stable" algorithms for highly unstable problems). We attempt to accomplish both goals at once by using the resulting algorithms to numerically compute eigenvalues for two well-known problems of Bessel and Hermite. These equations were chosen for the *sake of illustration* and chosen both because of their notoriety and the fact that they are equations of considerably different behavior than that of the equations previously treated in [1]–[3]. In particular, the Hermite and Bessel equations both have first derivative terms and the coefficients do *not* approach constants in the limit; moreover, Bessel's equation is singular on the interval (0, 1).

Finally, we indicate how the imbedding approach can be used to provide an algorithm for computing the asymptotic behavior of the eigenfunctions. This is done in Section V by obtaining an algorithm for computing the coefficient of the leading term of the Hermite polynomials.

For the sake of brevity many details and most of the rigor is omitted. The interested reader is referred to [3] with the encouragement that the analogous theorems can be obtained for the problems considered here (as well as many other problems). However, it is hoped that enough of the technique is illustrated to enable the interested person to apply the method to problems that he may be concerned with.

II. Some Preliminaries

We briefly consider the following general second-order initial-value problem and initiate the imbedding technique.

$$u'' + a(t)\,u'(t) + b(t)\,u = 0, \qquad u(x) = \cos\theta, \qquad u'(x) = \sin\theta \tag{2}$$

where a and b are continuous on some interval $\alpha \leqslant t < \beta \leqslant \infty$. We are interested in the behavior of the solution, $u(t; x, \theta)$, as $t \to \beta$.

Suppose, for fixed x and small Δ, we compare the two solutions to the corresponding initial value problems, $u(t; x, \theta)$ and $u(t; x + \Delta, \theta^+)$; where $\theta^+ = \theta^+(\Delta)$ is so chosen so that

$$u(t; x, \theta) = \eta u(t; x + \Delta, \theta^+). \tag{3}$$

Expressions for η and θ^+ can be obtained as follows. Using the mean-value theorem, we have

$$\begin{aligned} u(x + \Delta; x, \theta) &= u(x; x, \theta) + u'(\tilde{x}; x, \theta)\,\Delta \\ &= \cos\theta + \Delta \sin\theta + o(\Delta), \\ u'(x + \Delta; x, \theta) &= u'(x; x, \theta) + u''(\bar{x}; x, \theta)\,\Delta \\ &= \sin\theta - \Delta[a(x)\sin\theta + b(x)\cos\theta] + o(\Delta) \end{aligned}$$

as $\Delta \to 0$. Define

$$\eta \equiv \{[u(x+\Delta; x, \theta)]^2 + [u'(x+\Delta; x, \theta)]^2\}^{1/2}$$
$$= 1 + \Delta \sin\theta[\cos\theta(1 - b(x)) - a(x)\sin\theta] + o(\Delta).$$

Setting $t = x + \Delta$ in (3) we see that θ^+ must satisfy

$$u(x+\Delta; x, \theta) = \eta u(x+\Delta; x+\Delta, \theta^+) = \eta \cos\theta^+;$$

hence

$$\cos\theta + \Delta\sin\theta + o(\Delta) = \cos\theta^+\{1 + \Delta\sin\theta[\cos\theta(1-b) - a\sin\theta]\} + o(\Delta)$$

as $\Delta \to 0$. Solving for θ^+ we are lead to the desired expressions for θ^+ and η.

$$\begin{aligned} \theta^+ &= \theta + \Delta[\cos^2\theta(1 - b(x)) - a(x)\sin\theta\cos\theta - 1] + o(\Delta) \\ \eta &= 1 + \Delta\sin\theta[\cos\theta(1 - b(x)) - a(x)\sin\theta] + o(\Delta) \end{aligned} \tag{4}$$

as $\Delta \to 0$. The importance of (3) and (4) will be apparent in the next section.

Let $u_1(t)$ and $u_2(t)$ be two solutions to the above differential equation such that $W(t) =$ Wronskian $[u_1, u_2] \neq 0$. It is well-known that we can express

$$u(t; x, \theta) = \{[\cos\theta u_2'(x) - \sin\theta u_2(x)]/W(x)\}\, u_1(t)$$
$$+ \{[\sin\theta u_1(x) - \cos\theta u_1'(x)]/W(x)\}\, u_2(t) \tag{5}$$

$$\equiv A(x, \theta)u_1(t) + B(x, \theta)u_2(t). \tag{6}$$

If the asymptotic behavior of $u_1(t)$ and $u_2(t)$, as $t \to \beta$, is known, then clearly the coefficients A and B will determine the asymptotic behavior of $u(t; x, \theta)$. The idea then is to obtain algorithms for computing A and B which do *not* depend upon knowing the solutions u_1, u_2 or $u(t; x, \theta)$. In most cases of interest one of the fundamental solutions, say $u_1(t)$, will be dominant as $t \to \beta$; in this situation we usually confine our interest to the coefficient A (the exception to this is Section V).

Before going further with the imbedding technique, it is necessary that one knows the asymptotic behavior of two fundamental solutions $u_1(t)$ and $u_2(t)$. This information can often be obtained from the classical power-series technique (applied at β) or the standard asymptotic behavior results for linear systems (e.g., see [4]) or perhaps from knowledge of the physical problem.

In the sequel we derive algorithms for computing one or both of the coefficients A and B for the Hermite and Bessel equations. These examples hopefully will illustrate rather clearly how the method can be applied to other problems.

III. The Hermite Equation

We now consider the following form of the Hermite problem

$$u'' - (2t/\lambda)\, u' + u = 0, \qquad u(x) = \cos\theta, \qquad u'(x) = \sin\theta \tag{7}$$

where $\lambda > 0$ is a parameter. We are especially interested in the behavior of the solutions, $u(t; x, \theta)$, as $t \to +\infty$.

It is easily verified that there exists two solutions to the Hermite equations as follows:

$$\begin{aligned} u_1(t) &= \frac{\lambda}{2} t^{-\lambda/2-1} e^{t^2/\lambda}[1 + o(1)]; \quad u_1'(t) = t^{-\lambda/2} e^{t^2/\lambda}[1 + o(1)], \\ u_2(t) &= -t^{\lambda/2}[1 + o(1)]; \quad u_2'(t) = -\frac{\lambda}{2} t^{\lambda/2-1}[1 + o(1)] \end{aligned} \tag{8}$$

as $t \to \infty$. Referring to (5), (6), and (8) it is clear that

$$u(t; x, \theta) = A(x, \theta)\, u_1(t) + B(x, \theta)\, u_2(t) \tag{9a}$$

where

$$A(x, \theta) = [\sin\theta x^{\lambda/2}(1 + o(1)) - \cos\theta \frac{\lambda}{2} x^{\lambda/2-1}(1 + o(1))] e^{-x^2/\lambda} \tag{9b}$$

as $x \to \infty$. We now obtain a way of computing $A(x, \theta)$.

Noting the behavior of A for large x we are lead to define

$$\tilde{A}(x, \theta) = A(x, \theta)\, e^{x^2/\lambda}\, (x + 1)^{-\lambda/2}.$$

Observe that $\tilde{A}(0, \theta) = A(0, \theta)$ and that

$$\tilde{A}(x, \theta) = \sin\theta + o(1)$$

as $x \to \infty$. Finally, computational considerations lead us to make the final definition

$$\mathcal{A}(x, \theta) = \tilde{A}(x, \theta) - \sin\theta$$

and it is $\mathcal{A}(x, \theta)$ that we work on.

It follows easily from (3) that

$$A(x, \theta) = \eta A(x + \Delta, \theta^+)$$

and hence

$$\begin{aligned} &\exp[-x^2/\lambda](x + 1)^{\lambda/2}[\mathcal{A}(x, \theta) + \sin\theta] \\ &\quad = \eta \exp[-(x + \Delta)^2/\lambda](x + \Delta + 1)^{\lambda/2}[\mathcal{A}(x + \Delta, \theta^+) + \sin\theta^+]. \end{aligned}$$

Using this last identity and the expressions for η and θ^+ given by (4), we obtain the partial differential equation

$$\frac{\partial \mathcal{A}}{\partial x} + \frac{\partial \mathcal{A}}{\partial \theta}\left[\sin 2\theta \frac{x}{\lambda} - 1\right] = \mathcal{A}\left[\cos^2\theta \frac{2x}{\lambda} - \frac{\lambda}{2x+2}\right] + \left[\cos\theta - \frac{\lambda \sin\theta}{2x+2}\right]. \tag{10}$$

Furthermore, by (9b) and the definition of $\mathcal{A}$ we get the "boundary condition"

$$\lim_{x\to\infty} \mathcal{A}(x, \theta) = 0 \quad \textit{uniformly in } \theta. \tag{11}$$

It can be shown that problem (10)–(11) has a unique solution. A representation of the solution is given by the method of characteristics; namely,

$$\mathcal{A}(x, \theta) = \int_x^\infty \left[\frac{\lambda \sin\tilde{\theta}}{2s+2} - \cos\tilde{\theta}\right] \exp\left\{\int_x^s \left[\frac{\lambda}{2p+2} - \cos^2\tilde{\theta}\frac{2p}{\lambda}\right] dp\right\} ds, \tag{12}$$

$$\tilde{\theta}'(s) = \sin 2\tilde{\theta}(s/\lambda) - 1; \qquad \tilde{\theta}(x) = \theta.$$

Summarizing, for a given value of (x, θ) the corresponding solution to (7) can be expressed

$$u(t; x, \theta) = (\lambda/2)t^{-\lambda/2-1}e^{t^2/\lambda}[A(x, \theta) + o(1)]$$

as $t \to \infty$; where

$$A(x, \theta) = (x+1)^{\lambda/2}e^{-x^2/\lambda}[\mathcal{A}(x, \theta) + \sin\theta]$$

and $\mathcal{A}$ can be computed using (12) above.

IV. The Hermite Eigenvalue Problem

We now illustrate the computational aspects of the results of the previous section by studying the eigenvalue problem for the Hermite equation. One usually states this problem for the interval $(-\infty, \infty)$; however, because of the symmetry (in t) of the equation it suffices to consider the interval $(0, \infty)$. Hence we consider

$$u'' - (2t/\lambda)\, u' + u = 0, \qquad u(0) = \cos\theta, \qquad u'(0) = \sin\theta \tag{13}$$

for $\lambda > 0$. Also we will consider only $\theta = 0$ and $\theta = \pi/2$.

We can pose the eigenvalue problem as follows:

> Find the values of $\lambda > 0$ so that the solution to (13) has the property $u(t; 0, \theta)\, e^{-\epsilon t} \to 0$ as $t \to \infty$ for any $\epsilon > 0$.

From the work of the previous section it is clear that λ is an eigenvalue iff $A(0, \theta) = 0$ and this is true iff $\mathcal{A}(0, \theta) + \sin\theta = 0$. Hence, for fixed θ, the eigenvalues for (13) are precisely the zeros of the function

$$\bar{A}(\lambda; \theta) \equiv \mathcal{A}(0, \theta) + \sin\theta$$

and $\bar{A}(\lambda; \theta)$ is given by

$$\bar{A}(\lambda; \theta) = \sin\theta + \int_0^\infty \left[\frac{\lambda \sin\tilde{\theta}}{2s+2} - \cos\tilde{\theta}\right] \exp\left\{\int_0^s \left[\frac{\lambda}{2p+2} - \cos^2\tilde{\theta}\,\frac{2p}{\lambda}\right] dp\right\} ds$$
$$\tilde{\theta}'(s) = \sin 2\tilde{\theta}(s/\lambda) - 1; \qquad \tilde{\theta}(0) = \theta. \tag{14}$$

Using (14) we can compute the eigenvalues of (13) by the following simple two-step process:

(1) For fixed θ (specifically for $\theta = 0$ or $\theta = \pi/2$) one computes $\bar{A}(\lambda; \theta)$ for a discrete set of λ-values, thus producing a rough graph of $\bar{A}$ vs. λ. In this step a rather course λ-grid can be used; also the infinite integral in (14) can be approximated by a relatively short integral.

(2) Using the approximate zeros of $\bar{A}$ suggested by the graph one can iterate in order to locate the eigenvalues to several decimal places. Using the secant method, convergence to 5–6 significant figures is obtained in 4–5 iterations. Fortunately, in this problem (as well as several other problems) we found that the integral in (14) converges more rapidly as λ approaches an eigenvalue.

The computational results are illustrated in Table I. As is easily verified (since the eigenvalues and eigenfunctions are known for this problem) for $\theta = 0$ [hence $u(0) = 1$ and $u'(0) = 0$] the resulting eigenvalues are: 4, 8, 12,... . For $\theta = \pi/2$ [hence $u(0) = 0$ and $u'(0) = 1$] the eigenvalues are 2, 6, 10,... .

The major advantage this algorithm has over more conventional techniques (e.g., numerically integrating the equation (13) for a large t-interval) is its stabilizing effect. It is common knowledge how difficult it is to integrate equations like (13) for large t-intervals; particularly, in attempting to integrate an eigenfunction one finds that it is, for large t, overpowered by the larger solution of exponential behavior. In contrast the algorithm above exhibits very good computational behavior; and this is particularly true for λ at and near eigenvalues.

V. The Asymptotic Behavior of the Hermite Polynomials

Once the eigenvalues are located it is often of interest to consider the eigenfunctions. One can, of course, for each eigenvalue integrate (13) numerically to obtain (in this case) the Hermite polynomials. However, if it is necessary to know

TABLE I

CALCULATION OF EIGENVALUES FOR THE HERMITE PROBLEM

A.	$\theta = \pi/2$		$\theta = 0$	
	eigenvalues are 2, 6, 10, ...		eigenvalues are 4, 8, 12, ...	
	λ	$\bar{A}(\lambda)$	λ	$\bar{A}(\lambda)$
	1.8000000	.03565	3.8000000	−0.02399
	2.3000000	−0.04915	4.3000000	0.03418
	2.8000000	−0.09308	4.8000000	0.08480
	3.3000000	−0.12282	5.3000000	0.12464
	3.8000000	−0.13341	5.8000000	0.15002
	4.3000000	−0.12718	6.3000000	0.15701
	4.8000000	−0.10545	6.8000000	0.14182
	5.3000000	−0.06964	7.3000000	0.10140
	5.8000000	−0.02183	7.8000000	0.03422
	6.3000000	0.03486	8.3000000	−0.05857
	6.8000000	0.09583	8.8000000	−0.17184
	7.3000000	0.15464	9.3000000	−0.29494
	7.8000000	0.20280	9.8000000	−0.41014
	8.3000000	0.22982	10.3000000	−0.49144
	8.8000000	0.22356	10.8000000	−0.50427
	9.3000000	0.17135	11.3000000	−0.40677
	9.8000000	0.06175	11.8000000	−0.15398
	10.3000000	−0.11243	12.3000000	0.29395

B. ITERATION FOR EIGENVALUES USING SECANT METHOD AND STARTING INTERVALS INDICATED BY SIGN-CHANGES ABOVE

λ	$\bar{A}(\lambda)$	λ	$\bar{A}(\lambda)$
2.02151	−0.00371	4.00555	0.00068
1.99760	0.00041	3.99981	−0.00003
2.00001	−0.00000	4.00004	0.00001
2.00000	−0.00000	4.00000	0.00000
5.99168	−0.00099	7.98320	0.00331
5.99968	−0.00004	7.99868	0.00026
6.00003	0.00000	8.00000	0.00000
5.99999	0.00000		
9.97558	0.00931	11.96917	−0.03287
9.99720	0.00108	11.99553	−0.00482
10.00002	−0.00001	12.00007	0.00008
10.00000	0.00000	12.00000	0.00000

the eigenfunctions for large t this is difficult due to the instability of the problem. The imbedding technique has perhaps a little to offer here in that we can obtain an algorithm which gives the asymptotic behavior of the eigenfunction.

We now consider $B(x, \theta)$ in (9). If λ is an eigenvalue then $A(x, \theta) = 0$ and hence

$$u(t; x, \theta) = B(x, \theta)\, t^{\lambda/2}[-1 + o(1)]$$

as $t \to \infty$. Proceeding as we did in Section III we define

$$\mathscr{B}(x, \theta) = (x + 1)^{\lambda/2}\, B(x, \theta),$$

and obtain the boundary value problem

$$\begin{aligned} &\frac{\partial \mathscr{B}}{\partial x} + \frac{\partial \mathscr{B}}{\partial \theta}\left[\sin 2\theta \frac{x}{\lambda} - 1\right] = \mathscr{B}\left[\frac{\lambda}{2x + 2} - \sin^2 \theta \frac{2x}{\lambda}\right], \\ &\lim_{x \to \infty} \mathscr{B}(x, \theta) = \cos \theta \quad \textit{uniformly in } \theta. \end{aligned} \tag{15}$$

A fact of interest is that the differential equation for $\tilde{\theta}$ [see (12)] which appears in the application of the method of characteristics to (15) and (10) is the same equation which appears when the Prüfer (or polar coordinate) transformation is applied to $(u(s), u'(s))$. Specifically the Prüfer transformation is

$$\begin{aligned} \phi(s) &= \arctan[u'(s)/u(s)], \\ r(s) &= \{[u(s)]^2 + [u'(s)]^2\}^{1/2}. \end{aligned}$$

If u solves (13) it is easily verified that

$$\phi'(s) = \sin 2\phi(s)(s/\lambda) - 1$$

which is precisely the differential equation for $\tilde{\theta}$ in (14) and (16). This is useful to us since, for λ an eigenvalue, we know by (8) that $u'(s)/u(s) \to 0$ as $s \to \infty$. Hence $\tilde{\theta}(s) \to \arctan(0) = n\pi$ for some integer n (recall $(s, \theta(s))$ describes the characteristics along which $\mathscr{B}$ will be computed). Thus as $x = s \to \infty$ the solution to (15) along the characteristic, $\mathscr{B}$, approaches $\cos n\pi = \pm 1$.

Remark. We could have avoided (temporarily, at least) this discussion of the limit of $\tilde{\theta}$ by defining $\mathscr{B}$ analogous to our definition of $\mathscr{A}$ in Section III (note that $\mathscr{B}(x, \theta) - \cos \theta \to 0$ as $x \to \infty$). However, in actually computing $\mathscr{B}$ it is (as discussed below) expedient to "integrate backwards"; and if this is done knowledge of the limit of $\tilde{\theta}$ is necessary.

Solving (15) using the method of characteristics and the limit just described we get

$$\mathscr{B} = \pm \exp \int_x^\infty \left[\sin^2 \tilde{\theta} \frac{2s}{\lambda} - \frac{\lambda}{2s+2}\right] ds,$$
$$\dot{\tilde{\theta}} = \sin 2\tilde{\theta}(s/\lambda) - 1; \qquad \tilde{\theta}(x) = \theta. \tag{16}$$

where the sign is + if the integer n is even, − if n is odd. The value of n can be determined if $\tilde{\theta}$ is integrated for large s. The infinite integral in (16) can be shown to converge.

Theoretically then, once the eigenvalues have been located (by work of Section IV) one can find $\mathscr{B}$ by (16); and then $B(0, \theta) = \mathscr{B}(0, \theta)$ for $\theta = 0, \pi/2$ prescribes the behavior of the eigenfunctions for large t. However, numerically this algorithm does *not* enjoy the advantages of the algorithm for $\mathscr{A}$. Specifically, experience has shown that the integral in (16) is slow to converge. Moreover, this problem is complicated by the fact that, when λ is an eigenvalue, the $\tilde{\theta}$-equation in (16) is unstable for large s (this is easily seen by inspection of the differential equation for s large and $\tilde{\theta}(s) \approx n\pi$).

The problem in computing (16) can be helped to some extent by integrating the θ-equation backwards from s "large" to $s = 0$. In this regard we point out the following:

(1) We know that $\tilde{\theta}(\infty) = \lim_{s\to\infty} \tilde{\theta}(s) = n\pi$ for some (unknown) integer n. Notice that in (16) the $\tilde{\theta}$-equation and the expression for $\mathscr{B}(x, \theta)$ are both periodic in $\tilde{\theta}$ with period π. Hence if we set $\tilde{\theta}_c(\infty) = 0$ (i.e., $\tilde{\theta}_c(S) = 0$ for S large) and integrate the $\tilde{\theta}$-equation backwards we see that $\tilde{\theta}_c(s) \approx \tilde{\theta}(s) - n\pi$ and the corresponding computation of $\mathscr{B}$ is the desired approximation except for the sign.

(2) Recall that the sign of $\mathscr{B}$ is determined by the integer n. But this can now be determined from the relation $\theta = \tilde{\theta}(0) \approx \tilde{\theta}_c(0) + n\pi$; for example,

$$n = \text{nearest integer} \left[\frac{\theta - \tilde{\theta}_c(0)}{\pi}\right].$$

Using this approach and (16) we are able to compute $\mathscr{B}$ to 2–3 significant figures. To obtain higher accuracy it is doubtful if this algorithm is practical. So we do have a way of computing the asymptotic behavior of the eigenfunctions but it does not have the excellent stability properties enjoyed by the eigenvalue algorithms.

VI. Eigenvalues for Bessel's Equations

We now illustrate the applicability of the imbedding technique to the singular finite-interval problem by considering Bessel's equation

$$u'' + \frac{1}{t} u' + \left(\lambda - \frac{m^2}{t^2}\right) u = 0 \qquad u(x) = \cos\theta, \qquad u'(x) = \sin\theta \tag{17}$$

Again, it is easily shown and well known that Bessel's equation has two solutions of the form

$$\begin{aligned} u_1(t) &= t^{-m}\,[1 + o(1)]; \qquad u_1'(t) = -mt^{-m-1}\,[1 + o(1)], \\ u_2(t) &= \frac{1}{2m}\,t^m\,[1 + o(1)]; \qquad u_2'(t) = \frac{1}{2}\,t^{m-1}\,[1 + o(1)], \end{aligned} \tag{18}$$

as $t \to 0^+$. So solutions to (17) can be expressed

$$u(t; x, \theta) = A(x, \theta)\, u_1(t) + B(x, \theta)\, u_2(t),$$

where

$$A(x, \theta) = \tfrac{1}{2}x^m[\cos\theta + o(1)]$$

as $x \to 0^+$. We define $\mathcal{O}\!\mathcal{l}(x, \theta) = 2x^{-m}A(x, \theta) - \cos\theta$ and proceeding as before we obtain

$$\frac{\partial \mathcal{O}\!\mathcal{l}}{\partial x} + \frac{\partial \mathcal{O}\!\mathcal{l}}{\partial \theta}\left[\cos^2\theta\left(1 - \lambda + \frac{m^2}{x^2}\right) - \frac{1}{x}\sin\theta\cos\theta - 1\right]$$

$$= \mathcal{O}\!\mathcal{l}\left[\frac{\sin^2\theta - m}{x} - \sin\theta\cos\theta\left(1 - \lambda + \frac{m^2}{x^2}\right)\right] - \left(\sin\theta + \frac{m}{x}\cos\theta\right), \tag{19}$$

$$\lim_{x\to 0^+} \mathcal{O}\!\mathcal{l}(x, \theta) = 0 \qquad \textit{uniformly in } \theta.$$

Once more the method of characteristics leads to a solution;

$$\mathcal{O}\!\mathcal{l}(x, \theta) = \int_x^0 \left(\sin\tilde\theta + \frac{m}{s}\cos\tilde\theta\right)$$

$$\times \exp\left\{\int_x^s \left[\frac{1}{2}\sin 2\tilde\theta\left(1 - \lambda + \frac{m^2}{p^2}\right) - \frac{\sin^2\tilde\theta - m}{p}\right] dp\right\} ds \tag{20}$$

$$\tilde\theta'(s) = \cos^2\tilde\theta\left(1 - \lambda + \frac{m^2}{s^2}\right) - \frac{1}{2s}\sin 2\tilde\theta - 1; \qquad \tilde\theta(x) = \theta.$$

The usual Bessel eigenvalue problem is, of course,

> Find the values of $\lambda > 0$ so that the solution to (17) with $x = 1$, $\theta = \pi/2$ is bounded on [0, 1].

Clearly the eigenvalues are precisely the zeros of $\mathcal{O}\!\mathcal{l}(1, \pi/2)$ which can be computed using (20).

The first two eigenvalues for Bessel's equation of orders $m = 1, 2$, and 3 were computed using (20) and the procedure outlined in Section IV. The results are shown in Table II. The eigenvalues were computed to six significant figures. As in Section V the algorithm worked quite well; here, however, a certain amount of care must be exercised in integrating into the singularity at zero.

TABLE II

CALCULATION OF EIGENVALUES FOR THE BESSEL PROBLEM

A.

$m = 1$ eigenvalues are: 14.6820, 49.2185		$m = 2$ eigenvalues are: 26.3745, 70. 8499		$m = 3$ eigenvalues are: 40.7064, 95.2775	
λ	$\mathcal{O}(\lambda)$	λ	$\mathcal{O}(\lambda)$	λ	$\mathcal{O}(\lambda)$
10.000000	−0.09631	23.000000	−0.01029	34.000000	−0.00668
17.000000	0.02315	30.000000	0.00743	41.000000	0.00020
24.000000	0.06707	37.000000	0.01411	48.000000	0.00365
31.000000	0.06838	44.000000	0.01441	55.000000	0.00487
38.000000	0.04930	51.000000	0.01144	62.000000	0.00478
45.000000	0.02373	58.000000	0.00724	69.000000	0.00396
52.000000	−0.00345	65.000000	0.00303	76.000000	0.00280
		72.000000	−0.00062	90.000000	+0.00064
				97.000000	−0.00018

B. ITERATION FOR EIGENVALUES USING SECANT METHOD AND STARTING INTERVALS INDICATED BY SIGN-CHANGES ABOVE

λ	$\mathcal{O}(\lambda)$	λ	$\mathcal{O}(\lambda)$	λ	$\mathcal{O}(\lambda)$
15.34471	0.00868	27.06779	0.00168	40.77875	0.00005
14.56261	−0.00160	26.19552	−0.00045	40.70750	0.00000
14.68438	0.00003	26.38008	0.00001	40.70680	−0.00000
14.68194	−0.00000	26.37467	0.00000	40.70680	0.00000
14.68201	0.00000				
49.44768	−0.00067	70.96360	−0.00005	95.45217	−0.00002
49.21098	0.00005	70.85196	−0.00000	95.26589	0.00000
49.21851	0.00000	70.85064	0.00000	95.28105	−0.00000
49.21859	0.00000	70.85068	−0.00000	95.27831	−0.00000
49.21860	0.00000	70.85068	−0.00000	95.27759	−0.00000

VII. CONCLUSIONS

It should be clear that the methods illustrated here can be applied to a large number of eigenvalue problems of the classical Schrödinger and Sturm–Liouville type. It is apparent that the ideas can also be used on other problems which are concerned with the asymptotic behavior of solutions to second-order linear differential equations. The key requirement for applying these techniques is the knowledge of the asymptotic behavior of two independent solutions. It is hoped that the type of algorithm we have discussed has features which make it an attractive tool for such studies.

Currently work is being done in attempting to apply these techniques to nonlinear second-order equations and higher order linear equations and systems.

An ALGOL program designed to make the type of parameter computations discussed here is available on request. It has the feature that any number of new codes [depending on the coefficients $a(t)$ and $b(t)$ in $u'' + au' + bu = 0$] can easily be added.

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